

Violations of complementarity enable beyond-quantum nonlocality, distinguishability and cloning

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We consider the consequences of hypothetical violations of the principle of complementarity. For two-level systems, it is shown that any preparation violating complementarity enables the preparation of a nonsignalling box violating Tsirelson’s bound. Moreover, these “superquantum” objects could be used to distinguish a plethora of non-orthogonal quantum states and hence enable improved cloning protocols. For higher-dimensional systems the main ideas are briefly sketched.

Introduction The basic postulates and results of any physical theory are based on principles that are strongly supported by empirical evidence. The principle of conservation of energy, for example, is a major pillar on all areas of physics and implies deep limitations on human experience: it is impossible to construct a perpetual motion machine, or overperform Carnot’s heat engine. General relativity theory is governed by the equivalence principle and by the bound on the maximal speed of interactions given by the speed of light. These and other celebrated principles are often not only simple to understand, but very precisely stated, giving profound intuitions on the laws of nature.

In quantum theory, there is an ongoing search for one or more physical principles that could explain the bounds on quantum correlations. More precisely, even though quantum systems can beat classical bounds of Bell-like inequalities, it is known that there are limits to the violations of local realism attained by quantum objects. There exists theoretical constructions known as nonlocal boxes which can violate Bell-like inequalities more than quantum systems, without violating the principle of non-signalling or basic probability axioms. There are many different proposals of physical principles that try to explain such bounds on quantum correlations [1], but there is no general consensus on their success (see, for example, [2]).

For quantum systems, one major law is the principle of complementarity, based on the observation that certainty in the measurement of a fixed physical property precludes certainty in the measurement of a complementary one. In the double-slit experiment, complementarity is quantitatively expressed by the duality relation

$$D^2 + V^2 \leq 1, \quad (1)$$

where D is the path distinguishability and V the fringe visibility, verified by both empirical [5] and theoretical [6] methods. According to Feynman, the double-slit experiment [3] “has in it the heart of quantum mechanics; in reality it contains the only mystery” of the theory [4]. Applications can be found in Wheeler’s delayed choice experiment [7], which culminated in the concept of the quantum eraser [8]. Recently there is a growing interest in re-interpreting complementarity [9, 10] without, however, violating the empirical relation (1). The purpose of the present contribution is to consider the consequences hypothetical violations of complementarity [11]. We

first formulate the principle in a very simple and operational way, relating it to the empirical unpredictability of incompatible measurements. Then it is shown that any preparation violating the principle implies the possibility of creating deterministically preparations that violate Tsirelson’s bound. Moreover, these “superquantum” preparations can be used to distinguish and clone a plethora of non-orthogonal quantum states.

Operational theories We will work in the general formulation of *operational theories* (for more details, see [36]). An operational theory models mathematically a physical experiment in terms of primitive notions as preparations, measurements, outcomes and systems. More precisely, a preparation is a completely specified experimental procedure; a set of mutually exclusive preparations for an experiment forms then a set \mathcal{P} . In an experiment, a preparation $P \in \mathcal{P}$ is subjected to a measurement M , which is an element of a set \mathcal{M} of mutually exclusive measurements. This irreversible procedure gives some outcome k , which is one element of a set \mathcal{K} of mutually exclusive and exhaustive outcomes. The goal of any operational theory is to determine the probability $p(k|P, M)$, i.e., the probability that outcome k occurs given that we are performing the measurement M of the preparation P . Shortly, an operational theory is a specification $\{\mathcal{P}, \mathcal{M}, \mathcal{K}, p(k|P, M)\}$. For example, quantum theory is an operational theory where the preparations are given by density operators, measurements are given by observables and the probabilities are calculated through the rule $\text{Tr}(\rho E_k)$, where $\{E_k\}$ are elements of a Positive Operator-Valued Measure (POVM) associated with the observable M .

For a two-level system it is usual to work in the so-called Bloch vector representation. A preparation is fully specified by a three-dimensional Bloch vector with real components $\mathbf{r} = (r_x, r_y, r_z)$; for example, x , y and z are understood as orthogonal directions in space for the Stern-Gerlach apparatus and as the three independent polarization degrees of freedom in optical setups. A measurement on a two-level system has only two outcomes, which we will denote by ± 1 ; we will refer to this dichotomic measurement in direction $\hat{\mathbf{n}}$ as $\sigma_{\hat{\mathbf{n}}}$. It is an empirical evidence that the probabilities are calculated through the formula

$$p(\pm 1|\mathbf{r}, \sigma_{\hat{\mathbf{n}}}) = \frac{1}{2}(1 \pm \mathbf{r} \cdot \hat{\mathbf{n}}). \quad (2)$$

The mean value of a measurement in direction $\hat{\mathbf{n}}$ is simply

$\langle \sigma_{\hat{n}} \rangle = p(+1|\mathbf{r}, \sigma_{\hat{n}}) - p(-1|\mathbf{r}, \sigma_{\hat{n}}) = \mathbf{r} \cdot \hat{n}$. Notice that this kind of representation was common place in optics before the advent of quantum theory, where the elements of the Bloch vector are called Stokes parameters and the Bloch sphere is also called Poincaré sphere. Thus the Bloch vector representation does not rely on the quantum formalism. In other words, this representation itself is an independent operational model of a two-level system.

Complementarity Inspired by the uncertainty relations of Heisenberg [14], Bohr introduced in a series of lectures and essays [15] the so-called *principle of complementarity* (PC) [16], which establishes that evidence obtained under different experimental arrangements are complementary, in the sense that they cannot be unambiguously determined: the very means of acquiring information forbids us of having absolute knowledge or arbitrary precision of physical quantities for some preparations.

To motivate the discussion, let us imagine a scientist that never had contact with quantum theory and receives as a gift a Stern-Gerlach apparatus and a source of spin-(1/2) particles. This scientist observes that when she measures $\sigma_{\hat{n}}$ in different directions, the outcomes appear with some probabilities that depend on the preparation and on the directions that she chooses to measure. She observes also that some special arrangements of the preparation and the direction of measurement yields total predictability of outcomes. For example, if she prepares the particles's beam polarized in the z direction and then measures σ_z , the outcomes are totally determined. However, after trying a large number of possible different arrangements of preparations and measurements, one inevitable question will appear to her: "Why is it not possible to predict with certainty (probability 1 or 0) the outcomes of measurements in two different directions, for a fixed preparation?" There is in principle no rule that forbids her of obtaining, for some fixed preparation, the outcome +1 with probability one in two different directions. It is clear then that there is some physical law that forbids this perfectly legitimate situation. The basic empirical evidence is that if one measures $\sigma_{\hat{n}}$ in a fixed direction \hat{n} and obtains +1 with certainty implies that the outcomes of measuring $\sigma_{\hat{m}}$ in a different direction \hat{m} do not occur with total certainty.

The discussion will be restricted mostly to two-level systems in what follows. An extension to higher dimensions is sketched in the end of the text. For a two-level system, the principle of complementarity reads:

Principle 1. *For a fixed preparation, measurements of $\sigma_{\hat{n}}$ and $\sigma_{\hat{m}}$ in non-colinear directions \hat{n} and \hat{m} are not both predictable.*

By *predictable* we mean that the outcomes of the measurement are totally determined, i.e., one occurs with unit probability, implying the other have zero probability of occurrence. Thus, predictability means we can certainly know the result of measuring $\sigma_{\hat{n}}$. Principle 1 then states that predictability of a measurement in a certain direction precludes the predictability of a measurement in a different direction; in this sense

these different measurements are complementary. Let us see now how the PC imposes bounds on the Bloch vector:

Observation 1. *For a two-level system, a preparation with Bloch vector \mathbf{r} satisfies the principle of complementarity iff $r = \|\mathbf{r}\| \leq 1$. Equivalently,*

$$\langle \sigma_{\hat{n}_1} \rangle^2 + \langle \sigma_{\hat{n}_2} \rangle^2 + \langle \sigma_{\hat{n}_3} \rangle^2 \leq 1, \quad (3)$$

with $\mathbf{n}_1, \mathbf{n}_2$ and \mathbf{n}_3 orthogonal directions.

Proof: Appendix A.

Observation 1 is equivalent to the usual notion of complementarity for the Mach-Zender interferometer [20] expressed by the duality relation (1). It is also equivalent to Larsen-Luis complementarity relations [21] and bounds given by entropic uncertainty relations [22]. Moreover, it is noteworthy that the preparations respecting the PC correspond to the Bloch ball of preparations described by quantum theory.

It is clear that a violation of (3) could lead to negative or greater than one values for (2) for some directions of measurement, so we need to justify how to properly handle this situation. It is easy to see that many directions of measurement give true values of probabilities (2): it is just a matter of having $|\mathbf{r} \cdot \hat{n}| \leq 1$. When this condition is satisfied, there are no inconsistencies in the formulation. For simplicity, we will not consider the results of measurements in the problematic directions with $|\mathbf{r} \cdot \hat{n}| > 1$. The main reason is that it is not necessary - three non-colinear measurements of $\sigma_{\hat{n}}$ in allowed directions \hat{n} are enough to determine the Bloch vector \mathbf{r} , without violating any rule of probability theory. Moreover this last situation refers more to an empirical question than a theoretical one [47].

Transformations of preparations Before proceeding, we need to specify how to transform one preparation into another. We can greatly simplify the calculations that will appear through the introduction of the well-known operator

$$\rho(\mathbf{r}) = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma}), \quad (4)$$

where \mathbf{r} is the Bloch vector associated to the preparation and $\boldsymbol{\sigma}$ is a vector composed by the Pauli matrices. This operator is hermitean and unit-trace; as shown in Observation 1, a preparation respects the PC iff $r \leq 1$, which means that $\rho(\mathbf{r})$ is a positive operator and corresponds to a density matrix. If we identify a measurement $\sigma_{\hat{n}}$ with the operator $\boldsymbol{\sigma} \cdot \hat{n}$ then we can use the mathematical machinery of quantum operators to simplify our discussion. In order to see this, we define the projector $\Pi_{\hat{n}} = (1/2)(I + \hat{n} \cdot \boldsymbol{\sigma})$; in the optics literature these are known as Jones' matrices. It is trivial that $\Pi_{\hat{n}} + \Pi_{-\hat{n}} = I$ and $\boldsymbol{\sigma} \cdot \hat{n} = \Pi_{\hat{n}} - \Pi_{-\hat{n}}$. The rule (2) can then be rewritten as $p(\pm 1|\mathbf{r}, \sigma_{\hat{n}}) = \text{Tr}[\Pi_{\pm \hat{n}} \rho(\mathbf{r})]$. We will assume then that the set of allowed transformations are composed by standard completely-positive non-trace increasing linear maps over the operators $\rho(\mathbf{r})$. An operation over $\rho(\mathbf{r})$ will then induce an operation over \mathbf{r} corresponding to usual processes in a two-level system experiment. For example, local unitaries over

$\rho(\mathbf{r})$ correspond to rotations of the Bloch vector \mathbf{r} and the matrix $P(\mathbf{r}, \mathbf{r}') = \rho(\mathbf{r}) \otimes \rho'(\mathbf{r}')$ represents the addition of an extra two-level system with Bloch vector \mathbf{r}' , where \otimes is the kronecker product of the individual matrices. Similar translations of multipartite two-level systems operations in terms of Bloch vector operations can be found in [48]. Thus, for preparations respecting the PC, there is no deviation from standard predictions of quantum theory [49]. Our formulation can be seen in this equivalent way as an extension of quantum theory in order to consistently account for a violation of complementarity. More precisely, the rule $p(\pm 1|\mathbf{r}, \sigma_{\hat{n}}) = \text{Tr}[\Pi_{\pm\hat{n}}\rho(\mathbf{r})]$ by itself does not rule out negative operators $\rho(\mathbf{r})$, since for many directions of measurements $\sigma_{\hat{n}}$ the values $p(\pm 1|\mathbf{r}, \sigma_{\hat{n}})$ are genuine probabilities - and this is enough to make predictions about the system at hand. For two-level systems, Observation 1 shows that the PC is equivalent to imposing the postulate of positive operators $\rho(\mathbf{r})$ [23], while a violation of the principle would demand the abdication of this postulate and a legitimate use of negative operators to represent preparations. We observe that similar extensions of quantum theory in terms of non-positive operators have been employed to represent nonlocal boxes [24, 25], for the construction of efficient simulation schemes [26], toy models of quantum theory [27, 28] and more recently to locally extend quantum mechanics in the formulation known as “boxworld” [29, 30].

Nonlocal box creation We restrict our discussion to the standard scenario where two observers perform dichotomic measurements A_1 and A_2 (first observer) and B_1 and B_2 (second observer). Defining the Bell operator

$$\mathcal{B} = A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2, \quad (5)$$

it is well-known that assumptions of locality, realism and free-choice imposes the Clauser-Horne-Shimony-Holt (CHSH) bound [31] $|\langle \mathcal{B} \rangle| \leq 2$. The maximal violation attainable by quantum states and measurements is the so-called *Tsirelson's bound* $|\langle \mathcal{B} \rangle| \leq 2\sqrt{2}$ [32]. As shown by Popescu and Rohrlich [33], there are non-signalling probability distributions which violate Tsirelson's bound and some even reach the maximum algebraic value $|\langle \mathcal{B} \rangle| \leq 4$. These theoretical constructions can be studied in the framework of nonlocal boxes [34]. In [25] a representation of nonlocal boxes in terms of negative probabilities was obtained. Similarly to our approach, there are restrictions on the set of allowed measurements, in such a way that negative values of probabilities are discarded. For researchers used to quasiprobabilistic representations, there is no surprise: as quantum states necessarily display negative probabilities in order to be represented in a classical framework [35, 36], “superquantum” states display negative probabilities in order to be represented in a quantum framework. We use now these ideas and constructions in [24] to show that violations of complementarity allow the construction of nonlocal boxes violating Tsirelson's bound.

Theorem 1. *For a two-level system, any preparation violating the principle of complementarity enables the deterministic generation of a bipartite preparation that violates Tsirelson's bound.*

The full proof of this result [46] is given in Appendix A, but we briefly sketch the main ideas. By Observation 1, a preparation violating the PC has a Bloch vector \mathbf{r} with $r > 1$. Then using the equivalent representation (4), it is possible to generate deterministically the following bipartite preparation:

$$P = \frac{1}{2}[(1+r)P_{Bell+} + (1-r)P_{Bell-}], \quad (6)$$

where $P_{Bell\pm}$ are bipartite preparations displaying the same correlations of Bell states in quantum mechanics. The decomposition above was originally proposed in [24] in order to represent post-quantum nonlocal boxes and we show explicitly the set of measurements that enables to violate Tsirelson's bound $|\langle \mathcal{B} \rangle| \leq 2\sqrt{2}$ whenever $r > 1$. Hence, there is a deep link between bounds imposed locally by complementarity and bounds on nonlocal correlations.

Distinguishability and cloning Preparations violating the PC respect linearity and thus it is expected that some kind of no-cloning theorem still applies. Indeed this is the case, as can be seen by the following extension of the theorem:

Theorem 2. *Two preparations with Bloch vectors \mathbf{r} and \mathbf{r}' are jointly-clonable only if $\mathbf{r} \cdot \mathbf{r}' = \pm 1$*

Proof: Appendix A.

The equations $\mathbf{r} \cdot \mathbf{r}' = \pm 1$ are those of two affine hyperplanes that cross the interior of the Bloch ball, whenever at least one of the preparations \mathbf{r} or \mathbf{r}' violates the PC. Remarkably, there are still states that are not able to be jointly distinguished/cloned, suggesting fundamental limits even in the case of strong violations of physical principles.

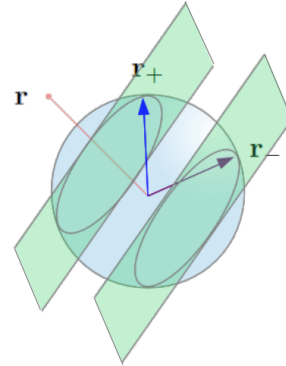


Figure 1. A preparation \mathbf{r} violating the PC (in red) defines two planes (in green) crossing the Bloch sphere (in light blue). These planes are formed by the preparations whose Bloch vectors satisfy $\mathbf{r} \cdot \mathbf{r}_{\pm} = \pm 1$.

The following result shows that an arbitrary preparation violating the PC can be used to distinguish some non-orthogonal quantum states. This enables naturally a protocol to clone these two states.

Theorem 3. *Given a preparation with Bloch vector \mathbf{r} violating the principle of complementarity, the quantum states with*

Bloch vectors \mathbf{r}_\pm satisfying $\mathbf{r} \cdot \mathbf{r}_\pm = \pm 1$ are distinguishable by a deterministic protocol.

The main idea to prove this result is to design a measurement where each outcome corresponds to a perfect correlation between \mathbf{r} and \mathbf{r}_+ and \mathbf{r} and \mathbf{r}_- exclusively. The full detailed proof is found in Appendix A.

Since we can discriminate with unit probability some non-orthogonal quantum states, the following is straightforward:

Corollary 1. *Given a preparation with Bloch vector \mathbf{r} violating the principle of complementarity, the quantum states with Bloch vectors \mathbf{r}_\pm satisfying $\mathbf{r} \cdot \mathbf{r}_\pm = \pm 1$ are clonable by a deterministic protocol.*

Proof: Appendix A.

For completeness, we refer the reader to other approaches to clone non-orthogonal states using closed time-like curves [42], which however rely on some form of nonlinear dynamics.

Higher dimensions For two-level systems it was shown that violations of the PC implies the possibility of violating Tsirelson's bound and in breaking the limits of distinguishability and cloning protocols. The distinctive feature in this situation was the relative independence on the typical rules associated to quantum theory, through the Bloch vector representation. Nevertheless, through (4) we argued that our formulation is equivalent to an extension of quantum theory in terms of non-positive operators for the preparations. We adopt this approach in order to formulate complementarity for higher-dimensional systems, i.e., we introduce an extension of quantum theory that does not impose positive-semidefiniteness on the operators representing preparations.

Explicitly, our “toy model” has the set of preparations \mathcal{P} composed of self-adjoint unit-trace operators $\tilde{\rho}$, the set of measurements \mathcal{M} composed of self-adjoint operators M and the probabilities are calculated via the trace-rule $p(k|\tilde{\rho}, M) = \text{Tr}(\tilde{\rho} E_k)$ where $\{E_k\}$ is the POVM associated to M ; it is noteworthy that within Hilbert-space formulations the trace-rule is unique [17]. Once again we consider only the results of measurements that give genuine values of probabilities $p(k|\tilde{\rho}, M)$. Let us introduce some definitions:

Definition 1. *Two non-degenerate measurements M and N are fully incompatible if they do not share any eigenstate.*

This definition captures the intuitive notion that a measurement is always disturbed if it is followed by a measurement that is fully incompatible with it, independent on the preparation that is measured.

Definition 2. *The outcomes of a non-degenerate observable are predictable if one of them occurs with unit probability.*

By *predictable* we mean that the outcomes of the measurement are totally determined, i.e., one occurs with unit probability, implying the others have zero probability of occurrence. Then we state our version of the PC for higher-dimensional systems:

Principle 2. *Given a fixed preparation, the outcomes of measurements of two non-degenerate fully incompatible measurements are not both predictable.*

This principle expresses the complementary aspect of fully incompatible measurements, since predictability of one quantity implies unpredictability of another quantity that is fully incompatible with it. It is easy to see that Principle 1 is a special case of Principle 2, when one uses the equivalent representation (4) and the identification $\sigma_{\hat{n}} \equiv \sigma \cdot \hat{n}$. Mathematically, given $P_\nu = |\nu\rangle\langle\nu|$, the principle simply states that there is no preparation $\tilde{\rho}$ for which $\text{Tr}(\tilde{\rho} P_\psi) = \text{Tr}(\tilde{\rho} P_\phi) = 1$, when ϕ and ψ are non-orthogonal.

Observation 1 shows that a two-level system preparation satisfies the PC iff the operator representing it is positive semidefinite. If this equivalence would hold as well for arbitrary dimensions, then complementarity would be the principle explaining the quantum bounds on non-local correlations, by the results of [25]. For higher-dimensional systems, however, this is not the case and the principle does not rule out all negative operators. To illustrate the main problems, let us consider a three-level system with orthonormal basis $\{|b_0\rangle, |b_1\rangle, |b_2\rangle\}$. The operator $\rho = (0.85)|b_0\rangle\langle b_0| + (0.25)|b_1\rangle\langle b_1| - (0.1)|b_2\rangle\langle b_2|$ is an example of non-positive operator that satisfies the PC, since its maximal eigenvalue is smaller than 1 and there is no rank-1 projective measurement for which the probability $\langle\psi|M|\psi\rangle$ is unit [44]. Hence, for higher-dimensional systems violation of the PC does not rule out completely preparations beyond quantum mechanics. Interestingly, preparations that do violate the PC as formulated here are still able to enhance the tasks of distinguishability and cloning in the same lines as the two-level case; the full argument is shown in Appendix B. Hence, violation of the PC is at least a necessary condition for performing beyond-quantum tasks.

Conclusions In this work, we gave a simple and operational formulation of the principle of complementarity in terms of the empirical unpredictability of fully incompatible measurements. For two-level systems it was shown that violation of complementarity is equivalent to: (i) the creation of non-local preparations that violate Tsirelson's bound without violating non-signalling, in the framework of the CHSH inequality, by using solely deterministic operations; (ii) distinguishability and hence cloning of a plethora of non-orthogonal quantum states via deterministic protocols. For higher-dimensional systems the equivalence does not hold, but violations of complementarity were shown as necessary for the enhancement of distinguishability and cloning. Thus, one can see our results as giving even stronger reasons for complementarity as a major physical principle and we believe it is, if not the main reason, one strong argument ruling out superquantum phenomena in nature. An important open problem is how to extend the formulation of the principle for higher-dimensional systems in order that only quantum preparations satisfy it, which is a theme we are currently working on.

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APPENDIX A - PROOFS OF RESULTS

Proof of Observation 1

If for a preparation with Bloch vector \mathbf{r} we have $r > 1$, i.e., the bound (3) is violated, writing $\mathbf{r} = r\hat{\mathbf{r}}$, we have that the probability of obtaining outcome +1 for the measurement $\sigma_{\hat{\mathbf{n}}}$ is

$$p(+1|\mathbf{r}, \sigma_{\hat{\mathbf{n}}}) = \frac{1}{2}(1 + r\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}) \quad (7)$$

It is easy to see that there exists an infinite number of unit vectors $\hat{\mathbf{n}}$ such that $\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} = 1/r$ and thus $p(+1|\mathbf{r}, \sigma_{\hat{\mathbf{n}}}) = 1$. Geometrically, this corresponds to the intersection between the affine plane $\mathbf{x} \cdot \mathbf{y} = 1/r$ and the unit sphere $\|\hat{\mathbf{x}}\| = 1$, which is satisfied by a circle where each point correspond to a direction $\hat{\mathbf{n}}$ such that $p(+1|\mathbf{r}, \sigma_{\hat{\mathbf{n}}}) = 1$. Thus, $\hat{\mathbf{n}}$ is fully predictable for an infinite number of non-colinear directions $\hat{\mathbf{n}}$. This proves the forward implication. Now, for the backward implication, if the PC is violated, we have that there exist a preparation $\hat{\mathbf{r}} = r\hat{\mathbf{r}}$ and non-colinear $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$ such that $\frac{1}{2}(1 + r\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}) = \frac{1}{2}(1 + r\hat{\mathbf{r}} \cdot \hat{\mathbf{m}}) = 1$, implying $r\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} = r\hat{\mathbf{r}} \cdot \hat{\mathbf{m}} = 1$. Thus, we must have $\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} = \hat{\mathbf{r}} \cdot \hat{\mathbf{m}} = 1/r$. Since $\hat{\mathbf{r}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$ are unit vectors and non-colinear, we have that $r = (\hat{\mathbf{r}} \cdot \hat{\mathbf{n}})^{-1} = (\hat{\mathbf{r}} \cdot \hat{\mathbf{m}})^{-1} > 1$, implying the bound (3) is violated and hence the backward implication and the

Observation are proven, QED.

Proof of Theorem 1

According to Observation 1, a preparation with Bloch vector \mathbf{r} violates the complementarity principle iff $r > 1$. Using the equivalent representation by the matrix (4), we have that $\rho(\mathbf{r})$ can be written in spectral decomposition as

$$\rho(\mathbf{r}) = \frac{1}{2}[(1+r)|\xi\rangle\langle\xi| + (1-r)|\xi^\perp\rangle\langle\xi^\perp|] \quad (8)$$

where ξ, ξ^\perp are the eigenstates of ρ . Defining vectors $|\pm_\xi\rangle = (1/\sqrt{2})(|\xi\rangle \pm |\xi^\perp\rangle)$ and the unitary

$$X_\xi = |\xi\rangle\langle\xi| - |\xi^\perp\rangle\langle\xi^\perp| \quad (9)$$

we apply the global unitary $U = |+\xi\rangle\langle+\xi| \otimes I + |-\xi\rangle\langle-\xi| \otimes X_\xi$ (basically a Controlled-NOT gate in a rotated basis) to the preparations $\rho(\mathbf{r}) \otimes |+\xi\rangle\langle+\xi|$, obtaining the bipartite preparation

$$P = U[\rho(\mathbf{r}) \otimes |+\xi\rangle\langle+\xi|]U^\dagger \quad (10)$$

$$= \frac{1}{2}[(1+r)|\phi_+\rangle\langle\phi_+| + (1-r)|\phi_-\rangle\langle\phi_-|] \quad (11)$$

where $|\phi_\pm\rangle = (1/\sqrt{2})(|+\xi+\xi\rangle \pm |-\xi-\xi\rangle)$. Define now the local unitary $U' = |0\rangle\langle+\xi| + |1\rangle\langle-\xi|$ and apply $U' \otimes U'$ to P above, obtaining

$$P' = \frac{1}{2}[(1+r)|\phi'_+\rangle\langle\phi'_+| + (1-r)|\phi'_-\rangle\langle\phi'_-|] \quad (12)$$

where $|\phi'_\pm\rangle = (1/\sqrt{2})(|00\rangle \pm |11\rangle)$.

For $r \leq \sqrt{2}$, let us choose $A_1 = (\sigma_x + \sigma_y)/\sqrt{2}$, $A_2 = (\sigma_x - \sigma_y)/\sqrt{2}$, $B_1 = \sigma_x$ and $B_2 = -\sigma_y$. It is easy to see that for these local measurements we have $\langle \mathcal{B} \rangle = Tr(\mathcal{B}P') = 2\sqrt{2}r$, i.e., a violation of Tsirelson's bound whenever $1 < r \leq \sqrt{2}$. For $r > \sqrt{2}$, we choose $A_1 = (\sigma_x + \sigma_y)/\sqrt{2}$, $A_2 = (\sigma_x - \sigma_y)/\sqrt{2}$, $B_1 = (\frac{\sqrt{2}}{r})\sigma_x + (\frac{\sqrt{r^2-2}}{r})\sigma_y$ and $B_2 = (\frac{\sqrt{r^2-2}}{r})\sigma_y - (\frac{\sqrt{2}}{r})\sigma_x$, obtaining $\langle \mathcal{B} \rangle = Tr(\mathcal{B}P') = 4$ for any value of r , i.e., the maximal violation of Tsirelson's bound that does not violate non-signalling. QED.

Proof of Theorem 2

Let the preparations with Bloch vectors \mathbf{r} and \mathbf{r}' be joint-clonable. Using (4), these preparations correspond to matrices $\rho(\mathbf{r})$ and $\rho'(\mathbf{r}')$. If these preparations are joint-clonable, then there exists an unitary U such that

$$U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger = \rho \otimes \rho \quad (13)$$

$$U(\rho' \otimes |e_0\rangle\langle e_0|)U^\dagger = \rho' \otimes \rho' \quad (14)$$

We then have

$$\text{Tr}[(\rho \otimes \rho)(\rho' \otimes \rho')] = \quad (15)$$

$$\text{Tr}[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger U(\rho' \otimes |e_0\rangle\langle e_0|)U^\dagger] = \quad (16)$$

$$\text{Tr}(\rho\rho') \quad (17)$$

where we used the cyclicity of trace in the last step. Since $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$, the first term is equal to $[\text{Tr}(\rho\rho')]^2$. Thus we have

$$[\text{Tr}(\rho\rho')]^2 = \text{Tr}(\rho\rho') \quad (18)$$

as a condition to existence of a unitary U that clones ρ and ρ' . This is equivalent to $\text{Tr}(\rho\rho') = 0$ or $\text{Tr}(\rho\rho') = 1$, which is equivalent to $\mathbf{r} \cdot \mathbf{r}' = \pm 1$, QED.

Proof of Theorem 3

Using (4), a preparation with Bloch vector \mathbf{r} is represented by the matrix

$$\rho(\mathbf{r}) = \frac{1}{2}(I + r\sigma \cdot \hat{\mathbf{r}}) \quad (19)$$

Let \mathbf{r}_+ and \mathbf{r}_- be the Bloch vectors of two quantum states such that $\mathbf{r} \cdot \mathbf{r}_\pm = \pm 1$. Let ρ_\pm be the matrices (4) representing the preparations with Bloch vectors \mathbf{r}_\pm ; then it is straightforward that $\text{Tr}(\rho\rho_-) = 0$, $\text{Tr}(\rho\rho_+) = 1$. These quantum states are expressible as

$$\rho_- = \frac{1}{2}[I - \frac{1}{r}(\sigma \cdot \hat{\mathbf{r}}) + y(\sigma \cdot \hat{\mathbf{m}}) + z(\sigma \cdot \hat{\mathbf{n}})] \quad (20)$$

$$\rho_+ = \frac{1}{2}[I + \frac{1}{r}(\sigma \cdot \hat{\mathbf{r}}) + y(\sigma \cdot \hat{\mathbf{m}}) + z(\sigma \cdot \hat{\mathbf{n}})] \quad (21)$$

where $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are directions orthogonal to $\hat{\mathbf{r}}$ and the real numbers y and z satisfy $(1/r^2) + y^2 + z^2 \leq 1$. Without loss of generality, we assume that $\hat{\mathbf{r}}$, $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ form a right-hand triple of vectors. We see that the Bell states are expressible as

$$|\phi_\pm\rangle\langle\phi_\pm| = \frac{1}{4}[I^{\otimes 2} \pm (\sigma \cdot \hat{\mathbf{r}})^{\otimes 2} \mp (\sigma \cdot \hat{\mathbf{m}})^{\otimes 2} + (\sigma \cdot \hat{\mathbf{n}})^{\otimes 2}]$$

$$|\psi_\pm\rangle\langle\psi_\pm| = \frac{1}{4}[I^{\otimes 2} \pm (\sigma \cdot \hat{\mathbf{r}})^{\otimes 2} \pm (\sigma \cdot \hat{\mathbf{m}})^{\otimes 2} - (\sigma \cdot \hat{\mathbf{n}})^{\otimes 2}]$$

Considering the probabilities of measurements on Bell basis for the states $\rho \otimes \rho_\pm$, we see that

$$\langle\phi_+|\rho \otimes \rho_-|\phi_+\rangle = 0; \quad \langle\phi_+|\rho \otimes \rho_+|\phi_+\rangle = 1/2 \quad (22)$$

$$\langle\psi_+|\rho \otimes \rho_-|\psi_+\rangle = 0; \quad \langle\psi_+|\rho \otimes \rho_+|\psi_+\rangle = 1/2 \quad (23)$$

$$\langle\phi_-|\rho \otimes \rho_-|\phi_-\rangle = 1/2; \quad \langle\phi_-|\rho \otimes \rho_+|\phi_-\rangle = 0 \quad (24)$$

$$\langle\psi_-|\rho \otimes \rho_-|\psi_-\rangle = 1/2; \quad \langle\psi_-|\rho \otimes \rho_+|\psi_-\rangle = 0 \quad (25)$$

Defining the projectors $P_\pm = |\phi_\pm\rangle\langle\phi_\pm| + |\psi_\pm\rangle\langle\psi_\pm|$, we see that they form a POVM, since $P_+ + P_- = I$. Moreover, from the results above it is straightforward that $\text{Tr}(P_+\rho \otimes \rho_+) = 1$ and $\text{Tr}(P_-\rho \otimes \rho_-) = 1$. Thus, if we have a state σ that is either ρ_+ or ρ_- - but we do not know which - we measure the POVM $\{P_+, P_-\}$ on the state $\rho \otimes \sigma$, obtaining the outcome 1 for P_+ (P_-) with unit probability iff σ corresponds to ρ_+ (ρ_-), QED.

Proof of Corollary 1

The preparation $\rho \otimes \sigma$ will be undisturbed by the measurement $\{P_+, P_-\}$, since the output from the protocol of Theorem 3 will be deterministically either $\rho \otimes \rho_+$ or $\rho \otimes \rho_-$. Notice that measuring $\sigma \cdot \hat{\mathbf{r}}$ and obtaining outcome $+1$ (-1) corresponds to projecting into P_+ (P_-). Since for an arbitrary state there is always a non-universal deterministic cloning protocol [43], after discriminating which state σ is, it is just a matter of applying the corresponding protocol to the output of the measurement $\{P_+, P_-\}$.

APPENDIX B - DISTINGUISHABILITY IN HIGHER DIMENSIONS

As explained in the main text, in order to violate the PC it is necessary that the operator ρ representing the preparation has at least one eigenvalue bigger than 1. Thus, an arbitrary preparation violating the PC in spectral decomposition reads

$$\tilde{\rho} = (1 + \epsilon)|\psi_0\rangle\langle\psi_0| + \sum_{k=1}^{d-1} \lambda_k |\psi_k\rangle\langle\psi_k| \quad (26)$$

where the $\{|\psi_n\rangle\}$ ($\{\lambda_n\}$) are the eigenvectors (eigenvalues) of $\tilde{\rho}$, ϵ is a positive real number and $\epsilon + \sum_{k=1}^{d-1} \lambda_k = 0$, implying $\text{Tr}\rho = 1$. Define an arbitrary pure state $|\nu_k\rangle = \sum_n \alpha_n^{(k)} |\psi_n\rangle$, with $\sum_n |\alpha_n^{(k)}|^2 = 1$. Then we have

$$\langle\nu_i|\tilde{\rho}|\nu_i\rangle = (1 + \epsilon)|\alpha_0^{(i)}|^2 + \sum_k \lambda_k |\alpha_k^{(i)}|^2 \quad (27)$$

For simplicity, let us consider first a vector $|\nu_1\rangle$ such that $|\alpha_1^{(1)}|^2 = |\alpha_2^{(1)}|^2 = \dots = |\alpha_{d-1}^{(1)}|^2$, implying $|\alpha_0^{(1)}|^2 + (d-1)|\alpha_1^{(1)}|^2 = 1$. Then $\langle\nu_1|\rho|\nu_1\rangle = 1$ is equivalent to

$$(1 + \epsilon)|\alpha_0^{(1)}|^2 + \left(\sum_{k=1}^{d-1} \lambda_k\right)|\alpha_1^{(1)}|^2 = 1 \quad (28)$$

$$(1 + \epsilon)|\alpha_0^{(1)}|^2 - \epsilon|\alpha_1^{(1)}|^2 = 1 \quad (29)$$

where we used the relation $\epsilon + \sum_{k=1}^{d-1} \lambda_k = 0$. Since $|\alpha_0^{(1)}|^2 + (d-1)|\alpha_1^{(1)}|^2 = 1$, one easily finds the solution

$$|\alpha_0^{(1)}|^2 = \frac{\epsilon + d - 1}{d\epsilon + d - 1} \quad (30)$$

and then an infinite number of vectors $|\nu_1\rangle$ such that $\langle\nu_1|\tilde{\rho}|\nu_1\rangle = 1$, i.e., such that the PC is violated by $\tilde{\rho}$. By the same reasoning, defining a vector $|\nu_0\rangle$ such that $|\alpha_1^{(0)}|^2 = |\alpha_2^{(0)}|^2 = \dots = |\alpha_{d-1}^{(0)}|^2$ but such that $\langle\nu_0|\tilde{\rho}|\nu_0\rangle = 0$ gives the solution

$$|\alpha_0^{(0)}|^2 = \frac{\epsilon}{d\epsilon + d - 1} \quad (31)$$

which is fulfilled by an infinite number of vectors as well.

Let us design then a POVM discriminating quantum states in the form $|\nu_0\rangle$ from those in the form $|\nu_1\rangle$. Define the following maximally entangled states

$$|\phi_k\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega_d^{jk} |\psi_k, \psi_k\rangle \quad (32)$$

where $\omega_d = e^{i(2\pi/d)}$ is the d -th root of unity; define the pro-

jector $P_1 = \sum_{k=0}^{d-1} |\phi_k\rangle\langle\phi_k|$. A straightforward calculation shows that $\text{Tr}(P_1 \tilde{\rho} \otimes |\nu_1\rangle\langle\nu_1|) = 1$ and $\text{Tr}(P_1 \tilde{\rho} \otimes |\nu_0\rangle\langle\nu_0|) = 0$. Thus, defining $P_0 = I - P_1$, we have a POVM $\{P_0, P_1\}$ such that $\text{Tr}(P_1 \tilde{\rho} \otimes |\nu_1\rangle\langle\nu_1|) = 1$ and $\text{Tr}(P_0 \tilde{\rho} \otimes |\nu_0\rangle\langle\nu_0|) = 1$, i.e., we can discriminate with certainty $|\nu_0\rangle$ from the (almost always) non-orthogonal $|\nu_1\rangle$ and by Corollary 1 in the main text, clone these states deterministically as well.